NEWTON AND THE INTERNATIONAL YEAR OF PHYSICS

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Abstract:
Following the logic used by Newton to drive his inverse-square law, a consistent theoretical model can be derived satisfying the modern tests for a theory of gravitation. Gravitational redshift is derived by applying special relativity as a kinematical effect. The model is compared with that of General Relativity (GR) and it is shown that GR follows as an approximation to this model.

Newton et L'Année Internationale de Physique

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Abstrait:
En poursuivant la logique utilisée par Newton pour aboutir à sa loi de l'inverse de la racine carrée, un modèle consistant peut être construit qui remplit les conditions requises des tests contemporains de la théorie de la gravitation. Le décalage spectral vers le rouge dû à la gravitation en est dérivé par l'application de la relativité spéciale en tant qu'effet cinématique. Le modèle est comparé avec celui de la Relativité Générale (RG) et il est montré que la RG apparaît comme une approximation de ce modèle.

Newton y el Año Internacional de la Física

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Extracto:
Siguiendo la lógica usada por Newton para derivar su inversa ley cuadrada, se puede derivar un modelo teórico consistente que satisfaga las modernas pruebas para una teoría de la gravitación. El desplazamiento al rojo cosmológico se deriva al aplicarse relatividad especial como un efecto cinemático. El modelo es comparado con el de la Relatividad General (RG) y se demuestra que la RG resulta como una aproximación a este modelo.

Newton e o ano internacional da física

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Sumário:
Seguindo a lógica usada por Newton para derivar sua lei do "quadrado inverso," um modelo teórico consistente pode ser derivado satisfazendo os testes modernos para uma teoria da gravitação. O "redshift" da gravitação é derivado pela aplicação da relatividade especial como um efeito cinemático. O modelo é comparado com aquele da Relatividade Geral (GR) e se demonstra que GR segue como uma aproximação a este modelo.

Newton und das internationale Jahr der Physik
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Zusammenfassung:
Nach der Logik welcher Newton in der Entwicklun seines Gesetzes des umgekehrten Quadrats folgte, kann ein theoretisches Modell geschaffen werden, dass einen modernen Test der Gravitationstheorie befriedigt. Gravitationelle Rotverschiebung ensteht durch den kinematischen Effect der speziellen Relativitaetstheorie Das Modell ist mit dem der allgemeinen Relativitaet (AR) verglichen und es wird gezeigt, dass AR als einer Annaeherung dieses Modells folgt.

NEWTON AND THE INTERNATIONAL YEAR OF PHYSICS

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1. INTRODUCTION

In 2005 we are celebrating the International Year of Physics. It coincides with the centenary of the three auspicious papers of Albert Einstein (1879–1955) published in that year. They were on the photo-electric effect (for which he received the Nobel prize for physics in 1921), Brownian motion and the theory of special relativity (SR). The latter theory dramatically changed the scientist's concept of time and space and heralded the conceptual division between the world of the scientist and that of the layman. In its wake followed discoveries in quantum physics, which further separated these two worlds. To the layman the world remained an orderly continuous sequence of events; to the physicist the discovery of Max Planck in 1900 of energy quanta showed that physical events can only occur in discrete steps.

Another dramatic change in the paradigms of scientific theories occurred in 1916 with Einstein's postulation of his General Theory of Relativity (GR). Not only was time and space interrelated, but they were related in a non-linear form. This form, known as the curvature of time-spatial coordinates in a generalized Riemannian space, gave an explanation for the subtle effects of gravitation.

One of these effects, the anomalous precession of the perihelion of Mercury, was first calculated by Leverrier in 1859. Not only do the planets move in ellipses about the sun, but the ellipses themselves are also rotating about the sun at extremely slow angular velocities.

General Relativity satisfactorily explained this precession. But it did more: It *predicted* that a massive object, such as the sun, would bend a light beam. It also predicted that in a gravitational field the spectrum of light would be shifted to lower frequencies, the so-called gravitational redshift.

The curvature of light was confirmed by Arthur Stanley Eddington (1882–1944) in 1919. To a world, weary of the destruction of the Great War, this confirmation of an astonishing prediction of the human intellect brought a relief that the human mind could transcend the boundaries of human conflict in a most practical way.

In this *Zeitgeist*, where the old order of nearly a thousand years of political and economic hegemony was abruptly destroyed, a fertile breeding ground of innovative scientific theories emerged. This same process was repeated thirty years later, when another cataclysmic war inaugurated the atomic, computer, synthetics and rocket age.

In the new scientific climate the ideas of Isaac Newton (1642–1727) were relegated to those of historical significance only. It also became vogue to compare the two great minds of science, Newton and Einstein, the absolutist and the relativist.[1]

To some extent such a comparison is unfair, as Newton had not been aware of the observations that had molded modern theories of gravitation. He was also limited in his mathematical techniques, in particular the mainstay of modern physics, the infinitesimal calculus. This necessary tool was only developed by him and Gottfried Wilhelm Leibniz (1646–1716) at that time.

It may be a fruitful exercise to turn back the clock and speculate on how Newton would have reasoned in modern times to derive a theoretical model of gravitation. As starting point we follow the logic he used to derive his theory of gravitation and adjust it to include modern observations. Newton's method of thought remains a universal one:

> "I keep the subject constantly before me, till the first dawnings open slowly, by little and little, into the full and clear light." [Isaac Newton entry in *Biographica Brittannica*, Vol. 5, London, 1760, pp. 32 – 41]

We shall be using several quotes by Newton and shall, where possible, provide their sources. An inventory of all the Newton papers is being prepared by the Newton Project. [2]

2. Modern Newton

It is not generally known that Newton first derived his inverse square law of gravitation by first considering circular orbits. [3,4] He applied Huygens's law for the acceleration in a circular orbit,

$$a = \frac{v^2}{r}, \tag{1}$$

and Kepler's third law to arrive at the inverse-square relation. He then proceeded to show in the *Principia* (there is some doubt about this[5]) that elliptical motion follows in general from this relation.

We extend this reasoning to incorporate modern results.

Today he would have been aware of the three classical tests for a theory of gravitation and that particles travelling near the speed of light obey relativistic mechanics. Following an iterative procedure he would have started with the simple model of circular orbits, derived the appropriate law of gravity but modified to accommodate relativistic effects, then generalised it to include the other conical sections. The model would finally be compared with all appropriate experimental results.

3. Finding a Lagrangian

For motion in a circular orbit under the gravitational attraction of a mass M one has:

$$\frac{v^2}{r} = \frac{GM}{r^2}. \tag{2}$$

Because of relativistic considerations, the ratio v^2/c^2 must be compared relative to unity, i.e.

$$\begin{aligned}(1 - v^2/c^2) &= 1 - GM/rc^2, \\ &= 1 - R/r,\end{aligned} \tag{3}$$

where

$$R = 2GM/c^2, \tag{4}$$

which is also known as the Schwarzschild radius.

Note that (2) is not an approximation of (3) for $v << c$. If we surmise that the inverse square law is only valid for $r >> R$, one could incorporate higher order gravitational effects by generalising the right-hand side of (3) to a polynomial:

$$\begin{aligned}1 - v^2/c^2 &= C(1 + a_1^{'}R/r + a_2^{'}R^2/r^2 + \ldots), \\ &= CP'(r),\end{aligned} \tag{5}$$

where C is a constant. This equation can be rewritten as

$$(1 - v^2/c^2)P(r) = C, \tag{6}$$

where

$$P(r) = 1 + a_1 R/r + a_2 R^2/r^2 + \ldots, \tag{7}$$

is the inverse of $P'(r)$.

In order to compare (6) with experiment, we have to convert it to some standard form in physics. To do this we first rewrite (6) as:

$$(1-C)c^2/2 = v^2/2 - GMa_1/r + a_1v^2R/2r + \ldots \tag{8}$$

If we multiply this equation by a constant, m_0, with the dimension of mass, we obtain a conservation equation with the dimensions of energy:

$$(1-C)m_0c^2/2 = m_0v^2/2 - GMm_0a_1/r + m_0a_1v^2R/2r + \ldots \tag{9}$$

For $r >> R$, this equation must approach the Newtonian limit:

$$m_0v^2/2 - m_0MGa_1/r = E_{N,} \tag{10}$$

where

$$E_N = (1-C)m_0c^2/2 \tag{11}$$

is the total Newtonian energy. Comparison with the Newtonian expression for the conservation of energy gives a_1=1 and the polynomial becomes

$$P(r) = 1 + R/r + a_2R^2/r^2 + \ldots$$

To simplify the notation, we define a constant E with dimensions of energy such that

$$C = E/m_0c^2. \tag{12}$$

Equation (6) can then be rewritten as

$$E = m_0c^2(1-v^2/c^2)P(r) = m_0c^2(1-v^2/c^2)(1+R/r+a_2R^2/r^2+\ldots). \tag{13}$$

There are now two approaches to determine the value of a_2. One may work with the polynomial $P(r)$ as it stands, derive the relevant equations of motion and compare the predicted results with the observed ones. One can then choose the value for a_2 that will make the predicted value agree with the observed value.

The other approach is to assume an initial value for a_2 and perform an iterative process until the predicted value fits the observed one. This is the approach that we shall follow.

Polynomials are cumbersome to work with and scientists always strive to find the simplest closed form, or function, for a polynomial. The form of $P(r)$ suggests various options but the first one that experience brings to mind is an exponential one. We therefore start with an inspired guess by letting a_2 =½, i.e. let

$$P(r) = 1 + R/r + R^2/2r^2 + \ldots = e^{R/r},$$
(14)

and see how this fits experimental results of higher order gravitational effects. The first value we look at is that of the observed perihelion precession. If our initial guess does not give the observed value we try an iterative procedure, starting with $a_2 = ½$, to find a predicted value as close as possible to the observed one. Newton is, in fact, the inventor of this procedure, which is named after him. We therefore rewrite (13) as

$$E = m_0 c^2 (1 - v^2/c^2) e^{R/r} = m_0 c^2 \frac{e^{R/r}}{\gamma^2},$$
(15)

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. If we consider (15) as the total energy of the system, we can find a corresponding Lagrangian by separating the potential and kinetic energies:

$$T = -m_0 v^2 e^{R/r},$$
$$V = m_0 c^2 e^{R/r}.$$
(16)

The signs for T and V are opposite to the conventional signs for these energies. This is due to the difference in the definitions of the total energy E and the conventional Newtonian energy E_N. For the latter, as can be seen from (10), the kinetic and potential energies have the conventional signs. This difference does not affect the outcome of the equations of motion.

The corresponding Lagrangian can be found from (16):

$$L = T - V = -m_0 (c^2 + v^2) e^{R/r}.$$
(17)

From this Lagrangian one can find the constants of motion of the system. For the conservation of energy, (15) follows again, while for the conservation of angular momentum we find

$$e^{R/r} r^2 \dot{\theta} = \text{constant} = h.$$
(18)

Equations (15) and (18) then give the quadrature of motion:

$$\frac{d\theta}{du} = \pm \left[\frac{e^{2Ru}}{h^2} - u^2 - \frac{Ee^{Ru}}{h^2} \right]^{-1/2},$$
(19)

where $u = 1/r$. Expanding the exponential terms to second degree in Ru yields a differential equation of generalized Keplerian form,

$$\frac{d\theta}{du} = (au^2 + bu + c)^{-1/2},$$
(20)

where

$$u = 1/r,$$
$$a = \left[R^2(4-E)/2h^2\right] - 1,$$
$$b = R(2-E)/h^2,$$
$$c = (1-E)/h^2.$$
(21)

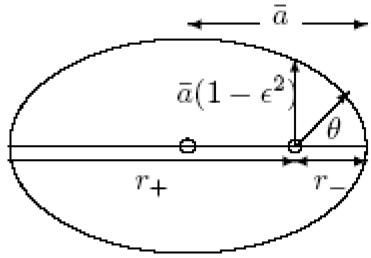

Figure 1: Elements of an ellipse

The convention $m_0 = c$ (velocity of light) = 1 was used.

Integrating (20) gives the orbit of a test particle as a generalized conic,

$$u = K(1 + \varepsilon \cos k\theta),$$
(22)

where the angles are measured from $\theta = 0$, and

$$k = (-a)^{1/2},$$
(23)

$$K = -\frac{b}{2a},$$
(24)

$\varepsilon = (1 - 4ac/b)^{1/2}$ = the eccentricity of the orbit.
(25)

4. Perihelion precession

In the case of an ellipse $\varepsilon < 1$, the presence of the coefficient k causes the ellipse not to be completed after a cycle of $\theta = 2\pi$ radians, i.e. the perihelion is shifted through a certain angle. This shift, or precession, can be calculated as (See Appendix A.1):

$$\Delta\varphi = 3\pi R/\bar{a}(1-\varepsilon^2),$$
(26)

where $\bar{a}$ is the semi-major axis of the ellipse. This expression gives the observed perihelion precession, confirming our choice of $a_2 = 1/2$. We must now see if this choice also satisfies the other classical test for a theory of gravitation, namely the bending of light by a massive object.

It is also useful to write the Lagrangian of (17) in terms of a general potential $\Phi = GM/r = Rc^2/2r$ as

$$L = -m_0(c^2+v^2)\exp(2\Phi/c^2).$$
(27)

5. Deflection of light

Newton has something specific to say about this:

> "Query 1. Do not bodies act upon light at a distance, and by their action bend its rays, and is not this action … strongest at the least distance?" [Newton, *Opticks* Query 1.]
>
> "And if Nature be most simple & fully consonant to herself she observes the same method in regulating the motions of smaller bodies (including the corpuscles of light) which she does in regulating those of the greater." [ULC Add. MS 3970. 3, fol. 336]

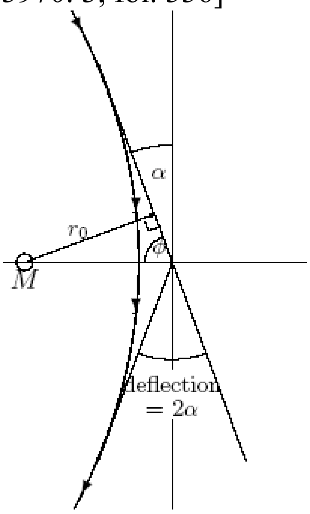

Figure 2: Deflection of light

We define a photon as a particle for which $v = c$. From (15) it follows that $E = 0$ and the eccentricity of the conic section is found to be (See Appendix A.2):

$$\varepsilon = r_0 / R, \tag{28}$$

where r_0 is the impact parameter. Approximating r_0 by the radius of the sun, it follows that $\varepsilon > 1$. The trajectory is therefore a hyperbola with total deflection equal to $2R/r_0$. This is in agreement with observation.

6. Gravitational Redshift

The above equations resulted from the application of a Lagrangian in which relativistic effects were not directly apparent. In the derivation of the gravitational redshift the relation of the above theory to special relativity becomes clearer. In order to find an expression for the gravitational redshift, one must accept the validity of special relativity at all points in physical space. This means that the time dilation formula

$$\gamma d\tau = dt \tag{29}$$

gives the *instantaneous* relation between the proper time $d\tau$ on a test particle moving about a central body and the coordinate time *dt* at that point in space with which the test particle is in coincidence. For example, in the rest frame of the Sun we have a distribution of coordinate clocks spread throughout the Solar System and all at rest relative to the Sun. The time intervals on a proper clock carried on a planet is compared with the time intervals measured on the *different* coordinate clocks in solar space as the planet passes them. This is a purely *kinematical relationship*, dependent on the relative velocities of the clocks only. The influence of the gravitational force does not explicitly appear in this relationship.

To determine the influence of gravity on the rate of clocks, we substitute for γ from (15) into (29):

$$\left(E / m_0 c^2\right)^{1/2} dt = \exp(R/2r) d\tau. \tag{30}$$

Defining an invariant frequency $\nu_0 = \left(E / m_0 c^2\right)^{1/2} / d\tau$ and $\nu = 1/dt$ we find

$$\begin{aligned} \nu &= \nu_0 \exp(-R/2r), \\ &\approx (1 - R/2r)\nu_0, \end{aligned} \tag{31}$$

which gives the observed gravitational redshift.

7. Special relativity

Equations (29) and (30) illustrate the respective kinematical and dynamical aspects of the theory. They also show how special relativity is incorporated in the model. The *dynamics* of the system, in this case gravitation, is responsible for the motion of a test particle. The *kinematics* of its motion, instantaneously at any point in space, is described by special relativity.

Special relativity is a theory of electrodynamics, and its explicit formulation in this model allows a synthesis of gravitation and electrodynamics.[6]

8. RELATION TO GENERAL RELATIVITY

If the Newtonian model gives the same results as GR for the classical tests, then it is to be expected that some relation should exist between the two models. As shown below, it turns out that GR is an approximation to the Newtonian model.

8.1 Brief overview of General Relativity

Since we shall frequently refer to the text by Weinberg[7], we shall refer to it by name where convenient.

Applying Einstein's field equations $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the Ricci tensor, to the standard form of the four-dimensional metric for a static, isotropic gravitational field,

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2,$$
(32)

gives the following values for the metric coefficients,

$$B(r) = 1 - R/r,$$
(33)

$$A(r) = (1 - R/r)^{-1}.$$
(34)

This form is known as the Schwarzschild metric. Applying the geodesics

$$\frac{d^2 x^\mu}{dp^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{dp} \frac{dx^\lambda}{dp} = 0,$$
(35)

to the Schwarzschild metric gives the constants of motion. [See (8.4.10) to (8.4.13) of Weinberg.]

$$r^2 d\theta / dp = J \text{ (angular momentum)},$$
(36)

$$-A(r)(dr/dp)^2 - J^2/r^2 + 1/B(r) = E \text{ (energy)},$$
(37)

$$dp = B(r)dt,$$
(38)

where the parameter p is normalized with respect to the coordinate t.

Eliminating p from (36) and (37), and letting $r = 1/u$ gives the quadrature of motion, or shape of the orbit,

$$\frac{d\theta}{du} = \pm\left[\frac{1-E}{J^2} + \frac{uRE}{J^2} - u^2 + Ru^3\right]^{-1/2}.$$
(39)

This differential equation differs from the Newtonian limit, or the Keplerian form of (20), by the presence of the Ru^3 term. In GR this term gives rise to the precession of the perihelion.

Extensions of $A(r)$ and $B(r)$ to generalized forms $\exp[\nu(r)]$ and $\exp[\lambda(r)]$ have been considered by Robertson and Noonan [8] and others. (See also the discussion by Weinberg in his section 8.3.)

9. GENERALIZED METRIC

As *Ansatz* we propose a *generalized metric*,

$$ds^2 = e^{-R/r}dt^2 - e^{R/r}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right).$$
(40)

We note that $A(r)$ and $B(r)$ are first-degree approximations to the exponential form,

$$\left(e^{R/r}\right)^{-1} = e^{-R/r} \approx \left(1 - \frac{R}{r}\right) = B(r) = A^{-1}(r).$$
(41)

Following the standard procedure of GR to determine the equations of motion (see par.8.4 of Weinberg), one finds the constants of motion,

$$\exp(R/r)r^2 d\theta/dp = h,$$
$$-\exp(R/r)(dr/dp)^2 - h^2\exp(-R/r)/r^2 + \exp(R/r) = E,$$
$$\exp(-R/r)dt = dp.$$
(42)

The corresponding equations of GR are (36), (37) and (38). The quadrature of motion can then be derived from the constants of motion given by (42) above:

$$\frac{d\theta}{du} = \pm\left[\frac{e^{2Ru}}{h^2} - u^2 - \frac{Ee^{Ru}}{h^2}\right]^{-1/2} .$$
(43)

This is the same as (19). We see that the Newton and generalized metric approaches yield the same equations of motion. It also follows that the equations of motion of GR are approximations to those of the modern Newtonian model.

The Newtonian approach also gives a simpler interpretation of gravitational redshift. The redshift arises from the difference in time intervals between two consecutive events measured at different gravitational potentials at the *same point in physical space*. From (40) this can be expressed as $ds = d\tau = \exp(-R/2r)dt$, where $d\tau$, the proper time, is the time interval measured in the absence of a gravitational field. Defining $\nu_0 = 1/d\tau$, $\nu = 1/dt$, we find

$$\nu = \nu_0 e^{-R/2r} \approx (1 - R/2r)\nu_0,$$
(44)

in accordance with the observed value for the gravitational redshift.

10. The Alchemist

In the modern flurry to 'publish or perish' it is informative to consider Newton's approach to science. Already in his time the rush for priority of publication was on, particularly in the *Transactions of the Royal Society*. Competition and jealousy were also as intense as now, as shown by the altercations between Newton and Hooke, as well as the polemic between Newton and Leibniz on the discovery of the infinitesimal calculus. Yet Newton seemed unconcerned about public praise:

> "For I see not what there is desirable in publick esteeme…"

and whose approach was for one purpose only:

> "When I wrote my treatise… I had an eye on such principles as might work with considering men for the belief of a Deity; and nothing can rejoice me more than to find it useful for that purpose."

However, there is ample evidence that Newton did succumb to bouts of vanity, but mainly to assert priority of his discoveries. Otherwise he could not be bothered. When Johann Bernoulli issued a challenge in 1697 to all the mathematicians in Europe to solve what is today known as the brachistochrone problem, he received an anonymous note, the only successful one, from England. Bernoulli recognized the author, prompting his classic response: "tanquam ex ungue leonem," or, "by the claw I recognize the lion." (See p.583 of ref. 3.)

Newton could also have anticipated a basic premise of quantum mechanics,

$mc^2 = h\nu$, arising from Einstein's 1905 paper on the photoelectric effect:

> "Are not gross bodies and light convertible into one another, … The changing of bodies into light, and light into bodies, is very conformable to the course of nature, which seems delighted with transmutation?"

That Newton had other motives in his pursuit of 'philosophy' has been given more prominence since the Keynes's collection of Newton's papers had become publicly accessible. Prior to that, a benign conspiracy existed to hide from public view Newton's preoccupation with alchemy, biblical history and the Temple of Solomon, amongst others. His papers on these esoterica far outnumber his work in conventional science. [9] (See also ref. 2.)

John Maynard Keynes referred to Newton as not the first of modern scientists, but the last of the magi.

11. Conclusion

The explicit incorporation of SR as a kinematical effect, resulting from the dynamics of the gravitational force, allows the incorporation of electrodynamics into the model. The resultant model can then be applied to find the spectrum of the hydrogen atom. It can also be extended to derive the Yukawa potential for the nuclear force. This unified model, the Holy Grail of modern physics, will be pursued in a separate paper.

In reaching this goal, Newton admonishes us to strive for simplicity:

> "It is the perfection of all God's works that they are done with the greatest simplicity… And therefore as they that would understand the frame of the world must endeavour to reduce their knowledge to all possible simplicity, so it must be in seeking to understand these visions." [JNUL, Yahuda MS7.21, fol.4.]

Since the GR metric of (32) is an approximation to the Newtonian one of (40), GR can be regarded as an approximation to the Newtonian theory. Both give the same predictions, but the Newtonian one is mathematically and conceptually simpler.

But reaching simplicity is not simple in itself. Newton has a word of warning for aspirant scientists, as applicable today as then:

> "Philosophy is such an impertinently litigious lady that a man had as good be engaged in law suits as to have to do with her." [*Correspondence*, Vol. 2, pp. 435–7]

And a final word from Newton:

> "Truth is the offspring of silence and unbroken meditation." [Keynes Ms. 130.7]

A. Appendix

A.1 Precession of the perihelion

After one revolution of 2π radians, the perihelion of an ellipse given by the conic of (22) shifts through an angle $\Delta\varphi = 2\pi / k - 2\pi$ or, from (23), as

$$\Delta\varphi = 2\pi\left[(-a)^{-1/2} - 1\right] ,$$
(45)

where a is given by (21). The constants of motion E and h are found from the boundary conditions of the system, i.e. $du / d\theta = 0$ at $u = 1 / r_-$ and $1 / r_+$, where r_+ and r_- are the maximum and minimum radii respectively of the ellipse. We find (see ref. 6):

$$E \approx 1 + R / 2\overline{a},$$
$$R^2 / h^2 \approx 2R / \overline{a}(1 - \varepsilon^2),$$
(46)

where $\overline{a} = (r_+ + r_-)/2$ is the semi-major axis of the approximate ellipse. Substituting these values in (21) gives

$$a = \frac{3R}{\overline{a}(1-\varepsilon^2)} - 1.$$
(47)

Substituting this value in (45) then gives (26).

A.2 Deflection of light

We first have to calculate the eccentricity ε of the conic for this case,

$$\varepsilon = (1 - 4ac / b^2)^{1/2}.$$

For a photon, setting $v = c$ in (21) gives

$$\varepsilon^2 = \left[-1 + \frac{h^2}{R^2}\right].$$
(48)

At the distance of closest approach, $r = r_0 = 1 / u_0$, we have $d\theta / du = 0$, so that from (19):

$$h^2 = e^{2Ru_0} / u_0^2 = r_0^2 e^{2R/r_0}.$$
(49)

From (48) and (49), and ignoring terms of first and higher order in R/r_0, we find

$$\varepsilon \approx \frac{r_0}{R}.$$
(50)

For a hyperbola $\cos\phi = 1/\varepsilon$, so that (see Figure 2)

$$\sin\alpha = 1/\varepsilon$$
$$\Rightarrow \alpha \approx 1/\varepsilon$$
$$\Rightarrow 2\alpha \approx 2R/r_0 = \text{total deflection}$$

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